VARIATIONS ON LUSIN'S THEOREM

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ABSTRACT. We prove a theorem about continuous restrictions of Marczewski measurable functions to large sets. This theorem is closely related to the theorem of Lusin about continuous restrictions of Lebesgue measurable functions to sets of positive measure and the theorem of Nikodym and Kuratowski about continuous restrictions of functions with the Baire property (in the wide sense) to residual sets. This theorem is used to establish Lusin-type theorems for universally measurable functions and functions which have the Baire property in the restricted sense. The theorems are shown (under assumption of the Continuum Hypothesis) to be "best possible" within a certain context.

1. Measurable functions. The results which appear here were announced at the Ninth Summer Symposium on Real Analysis, held at the University of Louisville during 1985, and appeared in abstract form (without proofs) in [9].

We study theorems about functions from a complete metric space X without isolated points (or else from the unit interval $I = [0,1]$) into the reals R. d will denote the metric for the space X. c denotes the cardinality of the continuum, and CH refers to the Continuum Hypotheses. A perfect set is a closed set M such that every point of M is a limit point of M, and a Cantor set is a homeomorphic image of the "middle thirds" Cantor subset of I.

The measurable functions we will be interested in are defined in terms of the following σ-algebras of subsets of X:

B_w: Baire property in the wide sense [22],

B_r: Baire property in the restricted sense [22],

L: Lebesgue measurable sets (assuming $X = I$),

U: Universally measurable sets (a set M is universally measurable if it is measurable with respect to the completion of every Borel measure on X),

(s): Marczewski measurable sets (a set M is Marczewski measurable provided that for every perfect subset P of X, there exists a perfect subset Q of P which either misses M or is a subset of M), and

B: Borel measurable.

The Marczewski measurable sets are most easily visualized as follows. Let the statement that a set M is "Bernstein dense" in a set P mean that M intersects every perfect subset of P. Then, a set M is Marczewski measurable or (s)-measurable provided there is no perfect set P in which both M and its complement are Bernstein dense (we would say that $M \cap P$ is one half of a "Bernstein subdivision of P" if both M and its complement were Bernstein dense in P). Property (s) for sets was defined by Marczewski in [25], where he established their basic properties and showed that the (s)-measurable functions were the same as the class of functions (studied by Sierpinski in [29]) f which are such that for every perfect set P, there exists a perfect set $Q \subset P$ such that $f \mid Q$ is continuous.

It is well known that these measurability properties are related to each other according to the following diagram of implications (assuming $X = I$ in the implication involving L):

(I)
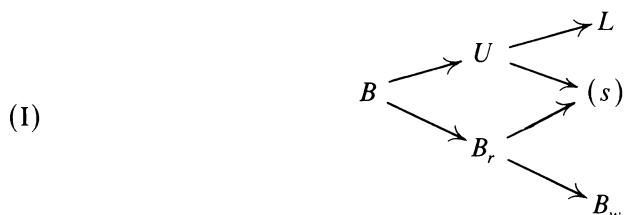
We will have occasion to refer to the σ-ideals associated with these σ-algebras:

FC: first category sets,

AFC: always first category sets [22],

L_0: Lebesgue measure zero sets (assuming $X = I$),

U_0: universal null sets (a set M is a universal null set provided it has measure zero relative to the completion of every continuous Borel measure on X),

(s^0): Marczewski null sets (a set M is a Marczewski null set provided it is true that for every perfect subset P of X, there exists a perfect subset Q of P which misses M), and

count: countable sets.

These singularity properties can be thought of as *hereditary* B_w, B_r, L, U, (s), and B, respectively, and therefore fit into the following diagram of implications (assuming $X = I$ in the implication involving L_0):

(II)
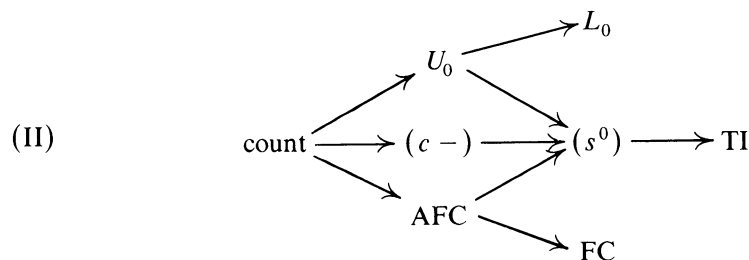
$(c-)$ means of cardinality less than c, and TI or "totally imperfect" means that the set contains no perfect subset.

If P is one of the singularity properties in the above diagram, we will say that a set D is "non-P dense in an open set J" if every open subset of J intersects D in a non-P set. However, we will say that D is "uncountably dense in J" rather than "non-count dense in J", that D is "categorically dense in J" rather than "non-FC dense in J", that D is "c-dense in J" rather than "non-$(c-)$ dense in J", and that D is "perfectly dense in J" rather than "non-TI dense in J", since this latter phenomenon occurs only if every open subset of J intersects D in a set which contains a perfect set.

We are interested in the following categoric notions of "bigness" of subsets D of X:

(1) D residual in X (i.e., $X - D$ is FC),

(2) D categorically dense in X,

(2′) D perfectly dense in X,

(3+) D c-dense in X,

(3) D uncountably dense in X,

(4+) D of cardinality $\geqslant c$ and dense in X.

(4) D uncountable and dense in X, and

(5) D dense in X.

We know these notions are related as follows,

(III)
$$(1) \longrightarrow (2') \longrightarrow (3+) \longrightarrow (4+)$$
$$(2) \longrightarrow (3) \longrightarrow (4) \longrightarrow (5)$$
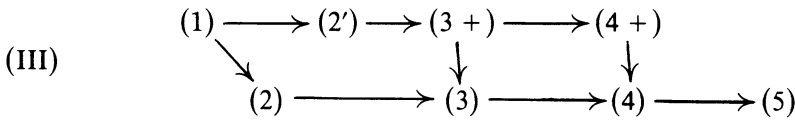
We are also interested in the following measure theoretic notions of "bigness" of subsets D of I:

(6) D of outer measure 1,

(7) D non-L_0 dense in I,

(8) D non-L_0 and dense in I,

(9) D non-L_0, and

(6+), (7+), (8+), and (9+), are the same as (6) through (9), respectively, except that D is assumed to be measurable.

We know these notions are related as follows:

(IV)
$$(6+) \longrightarrow (7+) \longrightarrow (8+) \longrightarrow (9+)$$
$$(6) \longrightarrow (7) \longrightarrow (8) \longrightarrow (9).$$
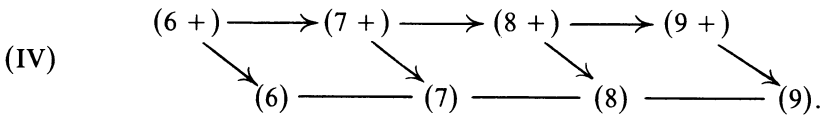
Lusin presented what has come to be known as "Lusin's theorem" in 1922 [24]. The theorem was actually know to be true by earlier researchers (the authors thank J. C. Morgan for bringing references [3, 23, and 32] to their attention). The following is a combination of Lusin's theorem and Blumberg's 1922 theorem [2]:

THEOREM 1. *For every L-measurable f: $I \to R$, there exists $D_1 \subset I$, D_1 of positive measure (9+), and $D_2 \subset I$, D_2 dense in I (5), such that both $f \mid D_1$ and $f \mid D_2$ are continuous.*

Nikodym proved the following category version of Lusin's theorem in 1929 [**28**]:

THEOREM 2. *For every B_w-measurable f: $I \to R$, there exists $D \subset I$, D residual in I (1), such that $f \mid D$ is continuous.*

Kuratowski [**21**] extended Theorem 2 to the metric case, where the range space is separable, and the case where the range space is nonseparable has recently been studied in [**12**] and [**13**].

It is well known that you can make the set D of Lusin's theorem have as large a positive measure less than 1 as desired, and that Lusin's theorem holds for any positive measure, but Theorems 1 and 2 are best possible in terms of having the set D satisfy one of the properties of (III) or (IV).

For example, it is well known that you cannot make the set D of Lusin's theorem satisfy (8) even for B-measurable f.

Nor can you make the set D of the Nikodym-Kuratowski theorem satisfy (9+) or (under CH) (9).

Ceder [**11**] recently gave an example which showed that you cannot make the set D of Lusin's theorem satisfy (4+) or (under CH) (4).

Ceder asked in [**11**], "Are there 'nice' kinds of functions, f, not having the property of Baire, for which there exists a dense subset D of I with D uncountable such that $f \mid D$ is continuous?" We show that the Marczewski measurable or (s)-measurable functions form such a class and that an even stronger result holds.

THEOREM 3. *For every (s)-measurable f: $X \to R$, there exists $D \subset X$, D perfectly dense in X (2'), such that $f \mid D$ is continuous.*

The proof of Theorem 3 will be given in §2, where the necessary tools concerning Marczewski sets will be developed, and it will be shown (under assumption of CH) in §3 that Theorem 3 is "best possible" in so far as making the set D satisfy one of the bigness properties of (III) or (IV) in that you cannot make the set D satisfy (2) (or even be non-FC) or (9).

It follows as a corollary to Theorem 3 that the following strengthened version of Lusin's theorem holds for U-measurable f.

THEOREM 4. *For every U-measurable f: $X \to R$, there exists $D_1 \subset X$ and $D_2 \subset X$, D_1 perfectly dense in X (2') and D_2 of positive measure (9+) (assuming $X = I$), such that $f \mid D_1$ and $f \mid D_2$ are continuous.*
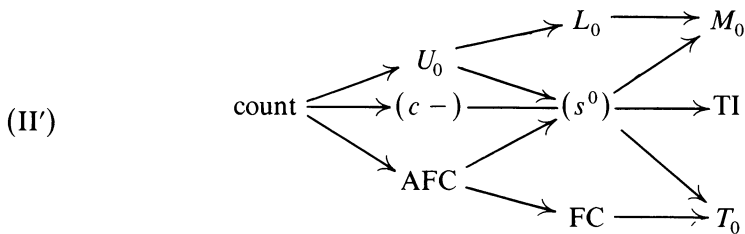
It is easy to show that you cannot find a single set D which will accomplish both jobs simultaneously in Theorem 4, even for B-measurable f. Under CH, it also follows that you cannot make the set D of Theorem 4 satisfy (1). In §3 it will actually be shown (under CH) that Theorem 4 is "best possible" in this context in that you cannot make the set D satisfy (2) or even be non-FC.

The following theorem is actually contained in the metric version of Theorem 2, but we state it separately for sake of reference.

THEOREM 5. *For every B_r-measurble f: $X \to R$, there exists $D \subset X$, D residual in X (1), such that $f \mid D$ is continuous.*

It is fairly easy to show (under CH) that you cannot make the set D of Theorem 5 satisfy (9+), but in §3 it will actually be shown (assuming CH) that Theorem 5 is "best possible" in this context in that you cannot make the set D satisfy (9).

2. Marczewski sets and proof of Theorem 3. We define two new singularity properties of sets, one of which will be used in the proof of Theorem 3. Let us say that a subset of I has property M_0 if it is the union of an L_0 set and an (s^0) set, and that a subset of X has property T_0 if it is the union of an FC set and an (s^0) set. We will actually only need the T_0 sets in our arguments but we state both definitions for the sake of symmetry. It is clear that the properties fit into (II) as follows (assuming $X = I$ for the implications involving L_0 and M_0):

(II′)

$$
\begin{array}{c}
\text{count} \rightleftarrows (c-) \rightrightarrows (s^0) \\
\end{array}
$$

(II′)   count ⇄ $(c-)$ ⇉ (s^0) ⟶ TI, with U_0, $L_0 \rightarrow M_0$, AFC, FC $\rightarrow T_0$.

We now state four lemmas concerning T_0 sets and arbitrary functions. These are analogous to the lemmas concerning FC sets and arbitrary functions which are usually employed in proving variations on Blumberg's theorem about continuous restrictions of arbitrary functions (see [1–8] and [33–34]).

LEMMA 1. *The T_0 sets form a σ-ideal of subsets of X, and no open subset of X is T_0.*

PROOF. They obviously form a σ-ideal because both the (s^0) sets and the FC sets do so. If an open subset J of X were the union of an FC set A and an (s^0) set B, there would be a perfect subset C of $J - A$, and then there would be a perfect subset D of C which missed B. This would be a contradiction.

LEMMA 2. *If Y is a non-T_0 subset of X, there exists an open subset Q of X such that Y is non-T_0 dense in Q.*

PROOF. This is the Banach category theorem for T_0 sets. Suppose that each open set Q has an open subset Q' such that $Q' \cap Y$ is T_0. Then, we can construct a collection $\{Q_\alpha\}$ of disjoint open subsets of X such that $Q_\alpha \cap Y$ is T_0 for each α and such that $\bigcup_\alpha Q_\alpha$ is dense in X. The complement of $\bigcup_\alpha Q_\alpha$ is FC and $\bigcup_\alpha (Q_\alpha \cap Y)$ is T_0, so it follows that Y is T_0.

REMARK. Lemmas 1 and 2 actually fit within Morgan's theory of category bases and follow from his fundamental theorem [27], but we present short proofs for completeness.

It is the need for this "Banach category theorem for T_0 sets" that caused us to introduce the T_0 sets in the first place. Notice that the analogous result with "T_0" replaced by "(s^0)" does not hold.

LEMMA 3. *Suppose Y is non-T_0 dense in X and $f: Y \to R$. Then there exists a $y \in Y$ such that for every neighborhood V of $f(y)$, there exists a neighborhood U of y such that $f^{-1}(V)$ is non-T_0 dense in U. In fact, the set E of all y in Y which do not have this property is a T_0 set.*

PROOF. This is analogous to Theorem II of [2], the Theorem of [1], and Lemma 3 of [4], with "FC" replaced by "T_0". Since we have Lemma 2 above, we can follow the proof of Lemma 3 of [4]. Let $\{G_n\}$ be a basis for R. For every n, let $Y_n = f^{-1}(G_n)$, let Q_n be the union of all open subsets of X in which Y_n is non-T_0 dense, and let $E_n = Y_n - Q_n$. It follows from Lemma 2 that the sets E_n are all T_0 sets, so $E = \bigcup_n E_n$ is also. But if the conclusion of the lemma fails for y, y is in E.

LEMMA 4. *Suppose Y is non-T_0 dense in X, $f: X \to R$ is (s)-measurable, and J is an open subset of X. Then there exists*
(1) *a Cantor subset C of J, and*
(2) *a subset Y' of Y such that Y' is non-T_0 dense in X such that $f\,|(Y' \cup C)$ is continuous at each element of C.*

PROOF. Assume without loss of generality that every $y \in Y$ has the property of Lemma 3.

$Y \cap J$ is not (s^0), so there exists a Cantor subset C_1 of J in which Y is Bernstein dense. f is (s)-measurable, so there exists a Cantor subset C_2 of C_1 such that $f\,|C_2$ is uniformly continuous. Y is also Bernstein dense in C_2. Let $e(1) > e(2) > \cdots$ be a sequence converging to 0 such that for each n, $e(n) > 0$ is such that for each x and y in C_2 with $d(x, y) < e(n)$, it follows that $|f(x) - f(y)| < 1/n$. For each y in $Y \cup C_2$ and each positive integer n, let $N(y, n)$ be a neighborhood (in X) of y of diameter $< e(n)/2$ such that the set $M(y, n) = f^{-1}[(f(y) - 1/n, f(y) + 1/n)] \cap N(y, n)$ is non-T_0 dense in $N(y, n)$. For each n, let $H(n) = \{N(y, n)\,|\,y \in Y\}$. Note that for each n, the set $V(n) = \bigcup H(n)$ contains all but at most countably many elements of C_2. If this were not the case, there would be a Cantor subset of $C_2 - V(n)$ which missed Y. Thus, $V = \bigcap_n V(n)$ also contains all but countably many elements of C_2. Therefore, there is a Cantor subset C of $V \cap C_2$. This will be the desired set C of the lemma.

We now inductively define a sequence $Y_1, Y_2, \ldots$ and a sequence $Z_1, Z_2, \ldots$, and the desired set Y' will be the union of the sets $Z_1, Z_2, \ldots$. Let $K(1)$ be a finite subcollection of $H(1)$ covering C and containing only sets which intersect C. Let $Z_1 = Y - \bigcup K(1)$, and let Y_1 be the union of the sets $M(y, 1)$ for which $N(y, 1)$ is in $K(1)$. Y_1 and Z_1 are mutually separated with union non-T_0 dense in X.

We now define Z_2 and Y_2. Let d_1 denote the minimum distance between a point of C and a point of the set $X - \bigcup K(1)$. Let n_2 be a positive integer such that $e(n_2) < d_1/2$. Let $K(2)$ be a finite subcollection of $H(n_2)$ covering C and containing only sets which intersect C. Let $Z_2 = Y_1 - \bigcup K(2)$, and let Y_2 be the union of the sets $M(y, n_2) \cap Y_1$ for which $N(y, n_2)$ is in $K(2)$. Z_1, Z_2, and Y_2 are mutually separated with union non-T_0 dense in X.

Continue this process. At the ith stage, $Z_1, \ldots, Z_i$, and Y_i will be mutually separated with union non-T_0 dense in X. Notice that if x is in C, z is in Y_i, and $d(x, z) < e(n_i)/2$, then $|f(x) - f(z)| < 2/n_i$. This is because z is in some set $N(y, n_i)$ which is in $K(i)$, and $|f(z) - f(y)| < 1/n_i$. x is within distance $e(n_i)/2$ of z, and z is within distance $e(n_i)/2$ of y, so x is within distance $e(n_i)$ of y, and $|f(x) - f(y)| < 1/n_i$ (x and y are both in C). It follows that $|f(x) - f(z)| < 2/n_i$.

It follows that the set $Y' = Z_1 \cup Z_2 \cup \cdots$ is non-T_0 dense in X and that $f|(Y' \cup C)$ is continuous at each element of C.

PROOF OF THEOREM 3. Since we are not assuming that X is separable, we will instead take advantage of the fact that there is a sigma-disjoint pseudo-base for X [33]. In fact, there exists a sequence $G_1, G_2, \ldots$ such that each G_n is a collection of disjoint open subsets of X, $\bigcup G_n$ is dense in X, each set in G_{n+1} is a proper subset of some set in G_n, and for each open subset J of X, there exists an n and an open set Q in G_n such that $Q \subset J$. For each open set J in G_1, let C_J be the Cantor subset of J and Y_J be the set Y' of Lemma 4 (letting $Y = X$) that goes with J (assume that C_J is nowhere dense in J). Let Y_1 be the union of the sets $Y_J \cap J$ such that J is in G_1 and let C_1 be the union of the sets C_J such that J is in G_1. Y_1 is non-T_0 dense in X, and $f|(Y_1 \cup C_1)$ is continuous at each element of C_1. Now, for each open set J in G_2, assume without loss of generality that J does not intersect C_1 and let C_J be the Cantor subset of J and Y_J be the set Y' of Lemma 4 (letting $Y = Y_1$) that goes with J (assume that C_J is nowhere dense in J). Let Y_2 be the union of the sets $Y_J \cap J$ such that J is in G_2 and let C_2 be the union of the sets C_J such that J is in G_2. Y_2 is non-T_0 dense in X, and $f|(Y_2 \cup C_2)$ is continuous at each element of C_2. Continue this process. It is clear that $D = C_1 \cup C_2 \cup \cdots$ is perfectly dense in X. $f|D$ is continuous. To see this, suppose $x \in D$ and $\varepsilon > 0$. Assume $x \in C_J$, where $C_J \subset C_n$ was constructed at stage n. Let $\delta > 0$ be such that if $y \in (C_J \cup Y_J)$ and $d(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon/2$. Also assume that δ is less than half the minimum distance from any point of C_J to a point of the complement of J. Now suppose $y \in D$ is such that $d(x, y) < \delta/2$. Then either $y \in C_J$ (in which case $|f(x) - f(y)| < \varepsilon$) or else $y \in C_{J'} \subset C_m$ for some $m > n$. In the latter case, y is within distance $\delta/2$ of some z in $Y_m \subset Y_n$ for which $|f(z) - f(y)| < \varepsilon/2$. But z will be a point of Y_n within distance δ of x, so $|f(z) - f(x)| < \varepsilon/2$. It follows that $|f(x) - f(y)| < \varepsilon$.

3. Examples.

In §1 it was stated several times that it is "fairly easy" or "possible" to describe an example based upon CH which shows that Theorems 3, 4, or 5 cannot be improved in some way or another. The constructions we had in mind on those occasions would have been based upon (1) the function $f: I \to R$ of Sierpinski and Zygmund [30] which has no continuous restriction of cardinality c, (2) the Lusin set (see [10]), and (3) the Sierpinski set (see [10]). We do not see how to piece together this function and these sets in a way which will accomplish what we desire to accomplish in this section. Therefore, we will describe transfinite construction processes somewhat more complicated than that of Sierpinski and Zygmund which will produce the desired result. We will use the following lemma in the construction of both examples.

LEMMA 5. *If $\{g_\alpha\}_{\alpha < \omega}$ is any countable collection of Borel measurable functions with domains Borel subsets of I, then there exists a Borel function f with domain I which misses $\bigcup_{\alpha < \omega} g_\alpha$.*

PROOF. Follows immediately from Corollary 4.3 of [**31**].

EXAMPLE 1. It follows from CH that there exists a U-measurable $f\colon I \to R$ which has no continuous restriction to a non-FC set.

PROOF. Notice that it follows from Lusin's theorem that a function $f\colon I \to R$ is U-measurable if and only if it is true that for every Borel measure μ on I there exists an F_σ FC set K such that $f \mid K$ is Borel. Let $\{x_\alpha\}_{\alpha < \Omega} = I$. Let $\{\mu_\alpha\}_{\alpha < \Omega}$ be a listing of the Borel measures on I. Let $\{D_\alpha, g_\alpha\}_{\alpha < \Omega}$ be a listing of the pairs D, g where D is a G_δ which is dense in some interval and $g\colon D \to R$ is Borel. We now define the steps in a transfinite inductive construction process.

Step 1. (1) Pick K_1 to be an F_σ FC set containing x_1 such that $\mu_1(K_1) = 1$ and (2) make $f \mid K_1$ be a Borel function which misses $g_1 \mid K_1$.

Step α. If $\mu_\alpha(\bigcup_{\beta < \alpha} K_\beta) = 1$, set $K_\alpha = \{x_\alpha\} - \bigcup_{\beta < \alpha} K_\beta$ and go on to the next step. Otherwise, (1) pick K_α to be an F_σ FC set disjoint from $\bigcup_{\beta < \alpha} K_\beta$ containing $\{x_\alpha\}$ (if $x_\alpha \notin \bigcup_{\beta < \alpha} K_\beta$) such that $\mu_\alpha(\bigcup_{\beta \leqslant \alpha} K_\beta) = 1$, and (2) make $f \mid K_\alpha$ be a Borel function which misses $\bigcup_{\beta \leqslant \alpha} (g_\beta \mid K_\alpha)$.

Part (1) of the construction process ensures that f will be defined on I and will be U-measurable.

Suppose S is a 2nd category set and $f \mid S$ is continuous. $f \mid S$ extends to a continuous function g which has domain a G_δ set D which is dense in some interval. D, $g = D_\beta$, g_β for some $\beta < \Omega$. S is 2nd category and $\bigcup_{\gamma \leqslant \beta} K_\gamma$ is FC, so there is an $\alpha > \beta$ such that $x_\alpha \in S - \bigcup_{\gamma \leqslant \beta} K_\gamma$. It follows that there is a first $\alpha > \beta$ such that K_α intersects $S - \bigcup_{\gamma \leqslant \beta} K_\gamma$ in a point x. But $f(x) = g_\beta(x)$ and $f \mid K_\alpha$ was chosen so as to miss $\bigcup_{\gamma \leqslant \alpha} g_\gamma \mid K_\alpha$. This is a contradiction.

EXAMPLE 2. It follows from CH that there exists a B_r-measurable $f\colon I \to R$ which has no continuous restriction to a non-L_0 set.

PROOF. Notice that a function $f\colon I \to R$ is B_r-measurable if and only if it is true that for every perfect set P, there exists a G_δ set K such that $K \cap P$ is dense in P (and therefore residual in P) such that $f \mid K$ is Borel. Let $\{x_\alpha\}_{\alpha < \Omega} = I$. Let $\{P_\alpha\}_{\alpha < \Omega}$ be a listing of the perfect subsets of I. Let $\{D_\alpha, g_\alpha\}_{\alpha < \Omega}$ be a listing of the pairs D, g where D is a G_δ of positive measure and $g\colon D \to R$ is Borel. We now define the steps in a transfinite inductive construction process.

Step 1. (1) Pick K_1 to be a G_δ of measure zero containing x_1 such that $K_1 \cap P_1$ is dense in P_1, and (2) make $f \mid K_1$ be a Borel function which misses $g_1 \mid K_1$.

Step α. If $(\bigcup_{\beta < \alpha} K_\beta) \cap P_\alpha$ is residual in P_α, set $K_\alpha = \{x_\alpha\} - \bigcup_{\beta < \alpha} K_\beta$ and go on to the next step. Otherwise, (1) pick K_α to be a G_δ subset of P_α of measure zero, disjoint from $\bigcup_{\beta < \alpha} K_\beta$, containing $\{x_\alpha\}$ (if $x_\alpha \notin \bigcup_{\beta < \alpha} K_\beta$) such that $(\bigcup_{\beta \leqslant \alpha} K_\beta) \cap P_\alpha$ is residual in P_α, and (2) make $f \mid K_\alpha$ be a Borel function which misses $\bigcup_{\beta \leqslant \alpha} (g_\beta \mid K_\alpha)$.

Part (1) of the construction process ensures that f will be defined on I and will be B_r-measurable.

Suppose S is a non-L_0 set and $f\,|\,S$ is continuous. $f\,|\,S$ extends to a continuous function g which has domain a G_δ set D which is of positive measure. D, $g = D_\beta$, g_β for some $\beta < \Omega$. S is of positive measure and $\bigcup_{\gamma \leqslant \beta} K_\gamma$ is L_0, so there is an $\alpha > \beta$ such that $x_\alpha \in S - \bigcup_{\gamma \leqslant \beta} K_\gamma$. It follows that there is a first $\alpha > \beta$ such that K_α intersects $S - \bigcup_{\gamma \leqslant \beta} K_\gamma$ in a point x. But $f(x) = g_\beta(x)$ and $f\,|\,K_\alpha$ was chosen so as to miss $\bigcup_{\gamma \leqslant \alpha}(g_\gamma\,|\,K_\alpha)$. this is a contradiction.

EXAMPLE 3. It follows from CH that there exists an (s)-measurable $f\colon I \to R$ which has no continuous restriction to either a non-FC set or a non-L_0 set.

PROOF. Let f_1 and f_2 be the functions of Example 1 and 2, and let M be an FC set of measure 1. Define $f(x)$ to be $f_1(x)$ if $x \in I - M$, and define $f(x)$ to be $f_2(x)$ if $x \in M$. f is the sum of a U-measurable function and a B_r-measurable function, so it is (s)-measurable, and f has no continuous restriction to either a non-FC set or a non-L_0 set.

PROBLEM. The examples given in this section rely heavily on use of CH. It would of course be preferable to obtain these examples without assuming CH. This might be a rather tall order. Even the existence of B_r sets which are not U sets, or U sets which are not B_r sets, is not known in ZFC, as far as we know. In considering this question in [25], Marczewski invoked CH and called upon the Lusin set as an example of a U_0 set which is not even B_w and the Sierpinski set as an example of an AFC set which is not even L-measurable. Indeed, the related examples given in the 1976 edition of Kuratowski and Mostowski's book [22] still rely on CH. Grzegorek and Ryll-Nardzewski [14–20] have recently made remarkable progress in obtaining related results in ZFC which were previously only known under CH, but these problems remain open as far as we know. See [10 and 26] for expository treatments of this subject.

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